Pergamon

Int. J. Heat Mass Transfer. Vol. 37, No. 16, pp. 2399–2404, 1994

0017-9310/94 $7.00+0.00

0017-9310(94)E0078-9

Centrifugally generated free convection in a rotating porous box

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(*Received* 23 *April* 1993 *and in final form* 11 *March* 1994)

Abstract—An analytical method of solution to the non-linear problem of centrifugally generated free convection in a rotating porous box is presented. The free convection results from differential heating of the horizontal walls leading to temperature gradients perpendicular to the centrifugal body force. The validity of the solution was found to be restricted to small values of the centrifugal Rayleigh number, although this restriction was not explicitly imposed. The results are compared to an asymptotic solution of the corresponding problem for a long rotating porous box. A direct extraction and substitution of the dependent variables was found to be useful in this case for de-coupling the non-linear partial differential equations, resulting in a set of independent non-linear ordinary differential equations which was solved analytically. The solution results, their significance and their validity domain are discussed in their physical context.

1. INTRODUCTION

FREE convection in rotating porous media is of practical interest in different engineering disciplines like the food process industry, the chemical process industry, centrifugal filtration processes and other modern, non-traditional applications of the porous media approach. A more detailed discussion regarding further applications is presented by Vadasz [1].

For a rotating porous matrix, the centrifugal acceleration is introduced as a body force. This force may create free convection in the same manner as the gravity force causes natural convection. For example, differential heating of the vertical walls generates unconditional natural convection under gravity conditions. Similarly, differential heating of the horizontal walls generates unconditional free convection under centrifugal conditions. Very few research results are available for natural convection in rotating porous media, most of them focusing on convection resulting from gravity in the presence of a single fluid or binary mixture (see Patil and Vaidyanathan [2], Jou and Liaw [3, 4], Rudraiah *et al.* [5], Chakrabarty and Gupta [6], and Palm and Tyvand [7]). The non-linear problem of free convection in rotating porous media resulting from the centrifugal body force was addressed by Vadasz [8, 9]. An asymptotic expansion was used by Vadasz [8] in the case of a small aspect ratio of the rectangular porous domain to obtain the solution for centrifugally generated free convection. Vadasz [9] presented the effect of Coriolis acceleration on the leading order free convection for high values of porous media Ekman number. Secondary flows in a plane orthogonal to the leading order free convection were obtained from the analytical solution of the governing equations.

It is common practice to rely on asymptotic solutions if they are confirmed by experimental measurements. As measurements related to the effect of rotation in porous media are not yet available, a comparison between the asymptotic solution and another analytical method becomes necessary in order to confirm the asymptotic solution results. A direct method of solution is presented in this paper and comparison between its results and the corresponding asymptotic solution is carried out. From the results it becomes evident that the present method is restricted to small values of the centrifugal Rayleigh number, although this restriction was not explicitly imposed.

2. PROBLEM FORMULATION

A rotating fluid saturated porous domain confined by rigid boundaries (Fig. 1) is considered. The rectangular domain is heated from above and cooled from below. Its lateral walls are insulated. The width

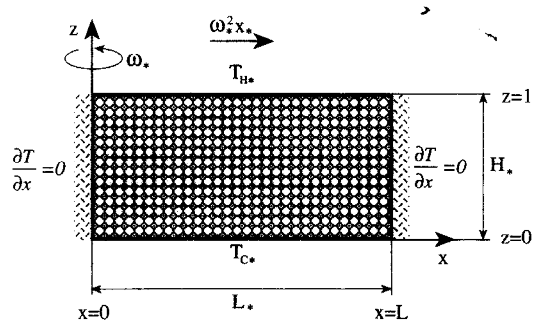

FIG. 1. A rotating rectangular porous domain heated from above and cooled from below. The subscript * stands for dimensional values.

NOMENCLATURE

$\hat{\mathbf{e}}_x$	unit vector in the x-direction
$\hat{\mathbf{e}}_y$	unit vector in the y-direction
$\hat{\mathbf{e}}_z$	unit vector in the z-direction
H_*	height of the domain, used as the length scale
k_*	permeability of the porous domain
L_*	length of the porous domain
L	aspect ratio, $= L_*/H_*$
M_f	ratio between the heat capacity of the fluid and the effective heat capacity of the porous domain
Nu	Nusselt number
p	reduced pressure generalized to include the constant component of the centrifugal term (dimensionless)
$\mathbf{q}$	dimensionless specific flow rate vector, equals $u\hat{\mathbf{e}}_x + v\hat{\mathbf{e}}_y + w\hat{\mathbf{e}}_z$
Ra_ω	porous media Rayleigh number related to the centrifugal body force, $= \beta_{T*}\Delta T_c \omega_c^2 H_*^2 k_* M_f/\alpha_{eo} v_0$
T	dimensionless temperature, $= (T_* - T_{C*})/(T_{H*} - T_{C*})$
T_{C*}	coldest wall temperature
T_{H*}	hottest wall temperature
u	horizontal x-component of the specific flow rate
v	horizontal y-component of the specific flow rate
w	vertical component of the specific flow rate
x	horizontal length coordinate
y	horizontal width coordinate
z	vertical coordinate.

Greek symbols

α_{eo}	effective thermal diffusivity
β_{T*}	thermal expansion coefficient
ΔT_c	characteristic temperature difference
v_0	fluid's kinematic viscosity
ψ	stream function
ω_c	angular velocity of the rotating box

Subscripts

*	dimensional values
0	reference values
c	characteristic values
C	related to the coldest wall
H	related to the hottest wall.

of the domain is much smaller than its height or length; thus the problem can be regarded as two-dimensional. Free convection occurs as a result of the centrifugal body force, while the gravity force is neglected. The solution is required for regions not close to the side-wall next to $x_* = L_*$. The Coriolis effect is considered small and most of the inertial forces are neglected as Darcy's regime is assumed. The only inertial effect considered is the centrifugal acceleration as far as changes in density are concerned. This assumption is compatible with the Boussinesq approximation for natural convection in pure fluids (non-porous domains). By assuming steady state conditions, the following set of governing equations is obtained:

$$\nabla \cdot \mathbf{q} = 0 \tag{1}$$

$$\mathbf{q} = -\nabla p - Ra_\omega x T \hat{\mathbf{e}}_x \tag{2}$$

$$\nabla^2 T - \mathbf{q} \cdot \nabla T = 0. \tag{3}$$

Equations (1), (2) and (3) are presented in a dimensionless form, where the values of H_*, $\alpha_{eo}/H_* M_f$, $\mu_* \alpha_{eo}/k_* M_f$ and ΔT_c are used to scale the length, specific flow rate, pressure and temperature variations, respectively. The centrifugal Rayleigh number is modified to include the centrifugal body force instead of gravity, in the form $Ra_\omega = \beta_{T*}\Delta T_c \omega_*^2 H_*^2 k_* M_f/\alpha_{eo} v_0$. The scaling temperature difference is chosen to be the difference between the hot and cold wall temperatures, thus $T = (T_* - T_{C*})/(T_{H*} - T_{C*})$. Following the assumption of two-dimensional flow, $v = 0$ and $\partial(\cdot)/\partial y = 0$. To obtain an analytical solution to the non-linear convection problem, a further assumption is made, namely that the vertical component of the specific flow rate w and the temperature T are independent of the horizontal coordinate x, i.e. $\partial w/\partial x = \partial T/\partial x = 0$ $\forall x \in (0, L)$ and are therefore functions of z only. It is this assumption that will subsequently restrict the validity domain of the results to small values of Ra_ω. Substituting these assumptions into equations (1)–(3) yields

$$\frac{\partial u}{\partial x} + \frac{dw}{dz} = 0 \tag{4}$$

$$u = -\frac{\partial p}{\partial x} - Ra_\omega x T \tag{5}$$

$$w = -\frac{\partial p}{\partial z} \tag{6}$$

$$\frac{d^2 T}{dz^2} - w\frac{dT}{dz} = 0. \tag{7}$$

3. ANALYTICAL SOLUTION

3.1. *Reduced set of equations*

The following method of solution is adopted for solving the non-linear coupled set of equations (4)–

(7). The temperature T is extracted from equation (5) and expressed in terms of x, $\partial p/\partial x$ and u, leading to

$$T = -\left[u+\frac{\partial p}{\partial x}\right]\Big/ Ra_\omega x.$$

This expression of T is introduced into equation (7) and the derivative $(\partial/\partial x)$ is applied to the result. Then, substituting the continuity equation (4) in the form $\partial u/\partial x = -\mathrm{d}w/\mathrm{d}z$ and equation (6) into the results and considering that $\mathrm{d}^2w/\mathrm{d}x^2 = 0$ yields an equation for w in the form:

$$\frac{\mathrm{d}^3 w}{\mathrm{d}z^3} - w\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = 0. \tag{8}$$

Equation (8) is a non-linear ordinary differential equation for $w(z)$. An interesting observation regarding equation (8) is the fact that it is identical to the Blasius equation for boundary layer flows of pure fluids (non-porous domain) over a flat plate. To observe this one simply has to substitute $w(z) = -f(z)/2$ to obtain $2f''' + ff'' = 0$, which is the Blasius equation. Unfortunately, no further analogy to the boundary layer flow in pure fluids exists, mainly because of the quite different boundary conditions and because the derivatives $[\mathrm{d}(\cdot)/\mathrm{d}z]$ and the flow (w) are in the same direction.

For u to be consistent with the assumption that $w \equiv w(z)$, it must be represented in the form:

$$u = x\varphi(z), \tag{9}$$

where $\varphi(z) = -\mathrm{d}w/\mathrm{d}z$ according to equation (4). To accommodate the equations and the assumptions, the pressure derivative $\partial p/\partial x$ in equation (5) must therefore have the form $\partial p/\partial x = xP$, where P is a constant, i.e. $w \equiv w(z) = -\partial p/\partial z$ and $T \equiv T(z)$, $u = x\varphi(z)$. To obtain the value of this constant P, a condition of no net flow through any vertical cross-section in the domain is imposed, stating that $\int_0^1 u\,\mathrm{d}z = 0$. By using equations (5) and (9), this condition implies that

$$\int_0^1 \varphi(z)\,\mathrm{d}z \equiv -\int_0^1 P\,\mathrm{d}z - Ra_\omega \int_0^1 T(z)\,\mathrm{d}z = 0; \tag{10}$$

hence we obtain a relationship between P and $T(z)$ in the form:

$$P = -Ra_\omega \int_0^1 T(z)\,\mathrm{d}z. \tag{11}$$

Therefore, the explicit numerical value of P should be obtained from the solution for T, w and u. An equation for $\varphi(z)$ is obtained similarly as equation (8) for w, by extracting w from equation (7), substituting it into the continuity equation (4) and using equations (5) and (6) after cross differentiating them, leading to

$$\varphi'\varphi''' - \varphi''^2 + \varphi\varphi'^2 = 0, \tag{12}$$

where $(\cdot)'$ stands for $\mathrm{d}(\cdot)/\mathrm{d}z$.

The relationship between $\varphi(z)$ and $T(z)$ is given by

$$T(z) = -\frac{1}{Ra_\omega}[P+\varphi(z)], \tag{13}$$

where P is a constant defined by (11).

In practice, solving equation (8) for $w(z)$ subject to the corresponding boundary conditions is sufficient for obtaining the complete solution. Following that, $\varphi(z)$ can be evaluated as $\varphi = -\mathrm{d}w/\mathrm{d}z$ [see text following equation (9)] and substituted into equation (13), once the value of P is known, to evaluate $T(z)$.

3.2. *Boundary conditions*

Three boundary conditions are necessary to solve equation (8) for $w(z)$. The non-penetration boundary conditions at the top and bottom, i.e. $w = 0$ at $z = 0$ and at $z = 1$, are two of the three. Additional boundary conditions are obtained by using the given temperatures at the boundaries, i.e. $T = 0$ at $z = 0$ and $T = 1$ at $z = 1$. Introducing these boundary conditions into (13) yields $\varphi(0) = -P$ at $z = 0$ and $\varphi(1) = -P - Ra_\omega$ at $z = 1$. Since $\varphi = -\mathrm{d}w/\mathrm{d}z$, the complete set of boundary conditions for w can be presented in the form:

$$z = 0: \quad w = 0 \quad \text{and} \quad \frac{\mathrm{d}w}{\mathrm{d}z} = P \tag{14a, b}$$

$$z = 1: \quad w = 0 \quad \text{and} \quad \frac{\mathrm{d}w}{\mathrm{d}z} = P + Ra_\omega. \tag{14c, d}$$

Equations (14a–d) represent four boundary conditions, while only three are necessary to solve equation (8). The reason for the fourth condition comes from the introduction of the constant P, whose value remains to be determined. Hence, the additional two boundary conditions are expressed in terms of the unknown constant P and the solution subject to these four conditions will determine the value of P as well.

3.3. *Method of solution*

A method similar to Blasius's method of solution is applied to solve equation (8). Therefore, $w(z)$ is expressed as a *finite power series* in the form:

$$w = \sum_{i=0}^{m} a_i z^i. \tag{15}$$

Substituting equation (15) into (8) and equating like powers of z leads to a set of recursive relationships among the coefficients. Substituting equation (15) into the boundary conditions (14) yields

$$a_0 = 0; \quad a_1 = P \tag{16a, b}$$

$$\sum_{i=1}^{m} a_i = 0; \quad \sum_{i=1}^{m} i a_i = P + Ra_\omega. \tag{16c, d}$$

The number of terms in the series (15) or (16c, d) was established from a balance between the requirement to have enough terms to get a significant polynomial expression for the third derivative in equation (8) and the need to keep the problem of evaluating the coefficients practical for solving by not including too many coefficients in the series. For $m = 7$ the recursive relationships among the coefficients can be solved to express the coefficients a_i ($i = 4, 5, 6, 7$) in terms of a_1 and a_2 in the forms $a_3 = 0$, $a_4 = a_1a_2/12$, $a_5 = a_2^2/30$, $a_6 = a_1^2a_2/120$, $a_7 = 11a_1a_2^2/1260$. Introducing these expressions and a_0 from (16a) into (16c, d) results in the following set of two non-linear algebraic equations for a_1 and a_2:

$$120a_1 + 120a_2 + 10a_1a_2 + 4a_2^2 + a_1^2a_2 + \frac{110}{105}a_1a_2^2 = 0 \tag{17a}$$

$$1240a_2 + 40a_1a_2 + 20a_2^2 + 6a_1^2a_2 + \frac{770}{105}a_1a_2^2 = 120Ra_\omega. \tag{17b}$$

It should be pointed out that according to (16b) the solution for a_1 determines the value of the constant P. The solution to the non-linear set of algebraic equations was found by using *Mathematica* [10] for symbolic as well as numerical computation. The de-coupling of equations (17a, b) leads to a sixth order algebraic equation for each of the unknowns a_1 and a_2. Since no analytical solution is known to the sixth polynomial order of an algebraic equation, the roots at this stage are obtained numerically. Accordingly, for any given value of Ra_ω, six sets of roots for $[a_1, a_2]$ are obtained. Some of the roots are complex or imaginary and are therefore ruled out, since $w(z)$ should be a real function. Among the remaining real roots the possibility of multiple solutions which are physically significant was investigated. The results show that other real roots have to be excluded as well, as they violate basic physical principles leaving the smallest set of roots as the only acceptable solution. These roots, when introduced into the expressions for the a_i values [see text following equation (16)] and into (15), transform expression (15) into the accurate solution of equation (8) and satisfy accurately the boundary conditions (14). Therefore, as long as this physically acceptable set of roots exists, equation (15) represents the analytical solution for $w(z)$. However, as the value of Ra_ω is increased more complex roots result from solving (17) and the physically significant roots disappear from the solution. It is at this value of Ra_ω where the validity of the solution breaks down. Including more terms in the power series representation of $w(z)$ may extend the range of Ra_ω values permitting acceptable solutions.

4. RESULTS AND DISCUSSION

The physically significant set of roots a_1 and a_2 are presented in Fig. 2(a) as a function of Ra_ω. The corresponding values of a_i ($i = 4, 5, 6, 7$) as evaluated by using the relationships among the coefficients [see text following equation (16)] are presented in Fig. 2(b, c). By comparing the curves in Fig. 2 it can be observed that while the higher order coefficients are small for

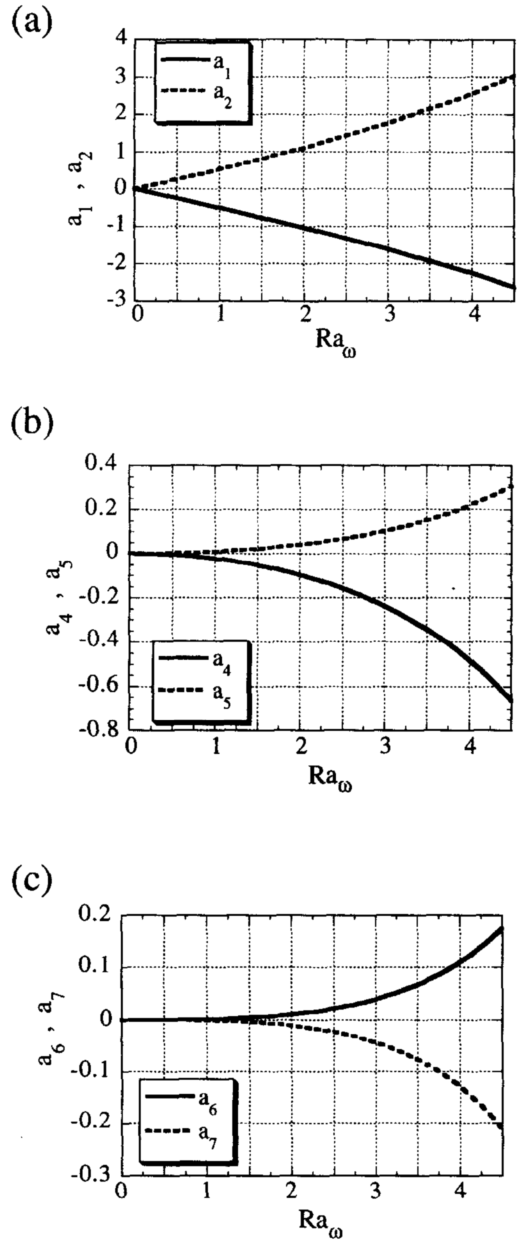

FIG. 2. Graphical description of the results for the coefficients as a function of Ra_ω: (a) values of a_1 and a_2; (b) values of a_4 and a_5; (c) values of a_6 and a_7.

small values of Ra_ω, their absolute values increase much faster than the values of a_1 and a_2 as Ra_ω increases. Therefore, for higher values of Ra_ω the higher order coefficients' contribution becomes more significant. Substitution of the evaluated coefficients into equation (15) yields the solution for $w(z)$. The solution for $\varphi(z) = -\mathrm{d}w/\mathrm{d}z$ as obtained upon taking the derivative of w and u is represented by equation (9), i.e. $u = x\varphi(z)$. Since according to (16a) $P = a_1$, the solution for T is evaluated next by using equation (13). These solutions are presented in Fig. 3 for two values of Ra_ω, i.e. $Ra_\omega = 0.5$ and $Ra_\omega = 4.5$. The solu-

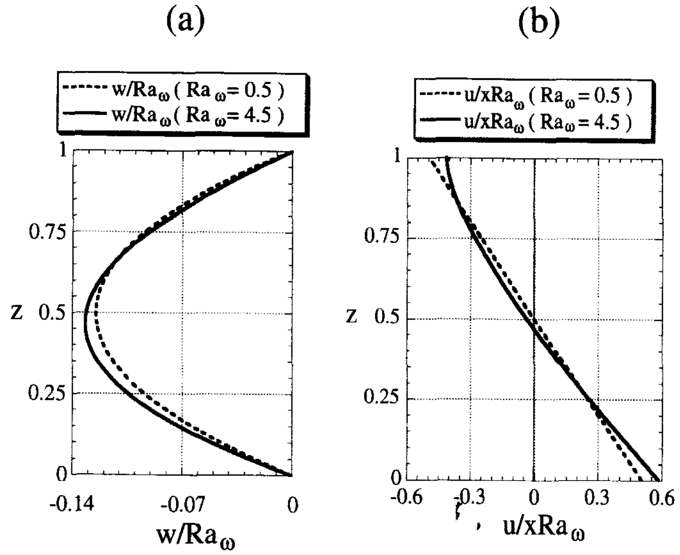

FIG. 3. Graphical description of the results for the specific flow rate components and temperature for two values of Ra_ω ($Ra_\omega = 0.5$ and $Ra_\omega = 4.5$). (a) Vertical profile of w/Ra_ω. (b) Vertical profile of u/xRa_ω. (c) Vertical temperature profile T.

tion for w/Ra_ω is presented in Fig. 3(a). It is apparent in the figure that only a slight variation of the values of $|w/Ra_\omega|$ results for $Ra_\omega = 4.5$ in comparison with $Ra_\omega = 0.5$. This suggests that the intensity of the flow varies almost linearly with Ra_ω. However, a more significant observation can be made from Fig. 3(a). The location where $|w|$ is maximum is $z = 0.5$ for $Ra_\omega = 0.5$, but this maximum moves downwards to $z = 0.47$ for $Ra_\omega = 4.5$, resulting in a break of symmetry as Ra_ω increases. The reason for this break of symmetry is better observed from Fig. 3(b), which represents the solution for u/xRa_ω. For $Ra_\omega = 0.5$, u/xRa_ω is a straight line representing a flow in the positive x-direction in the bottom half of the domain, i.e. for $z \in [0, 0.5]$ and a backward flow (in the negative x-direction) in the top half, i.e. for $z \in [0, 0.5]$. However, for $Ra_\omega = 4.5$ the zero of u/xRa_ω does not occur at $z = 0.5$ but at $z = 0.47$, and the plot is no longer linear. This effect, which results from increasing the value of the centrifugal Rayleigh number, is apparent in Fig. 3(c), which represents the temperature profile. The convection effect on $T(z)$ is clearly felt for $Ra_\omega = 4.5$, while for $Ra_\omega = 0.5$, $T(z)$ is linear pertaining to a conduction regime. The break of symmetry with increasing Ra_ω was also obtained in ref. [8] by using an asymptotic solution, although the results presented in ref. [8] are not definite in this aspect as the asymptotic expansion solution is limited to the first two terms in the series.

As the flow is two-dimensional, streamlines can be plotted by introducing the stream function corresponding to the flow solution, i.e. $u = \partial\psi/\partial z$ and $w = -\partial\psi/\partial x$. This stream function is introduced only for the purpose of presentation of results. Therefore, an example of the flow field represented by the streamlines is presented in Fig. 4 for $Ra_\omega = 4$ and for an aspect ratio of 3 (excluding the region next to the sidewall at $x = L$). Outside the end region next to $x = L$ the streamlines remain open on the right hand side. They are expected to close in the end region. Nevertheless, the streamlines close on the left hand side, throughout the domain. The reason for this is the centrifugal acceleration, which causes u to vary linearly with x, thus creating (due to continuity equation) a non-vanishing vertical component of the specific flow rate w at all values of x.

The local Nusselt number representing the local vertical heat flux is defined as $Nu = |-\partial T/\partial z|_{z=0}$. Substituting for T by using its previously presented solution, the value of Nu is expressed by $Nu = 2a_2/Ra_\omega$. By introducing the corresponding values of a_2 as presented previously for different values of Ra_ω into this relationship for Nu, its value is evaluated as a function of Ra_ω. The results of this evaluation are presented in Fig. 5. A comparison between the present results of Nu and the results obtained by Vadasz [8] using an asymptotic solution is also presented in Fig. 5. The two results compare well as long as Ra_ω is very small. However, for increasing Ra_ω the

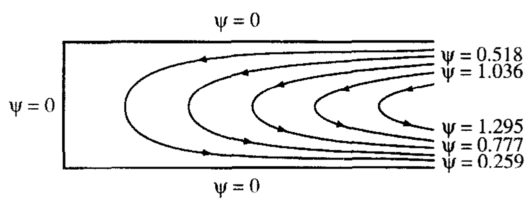

FIG. 4. Graphical description of the resulting flow field; five streamlines equally spaced between their minimum value $\psi_{min} = 0$ at the rigid boundaries and their maximum value $\psi_{max} = 1.554$. The values in the figure correspond to every other streamline.

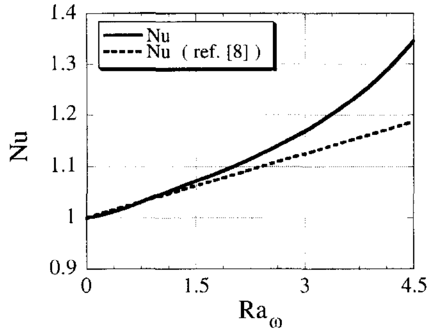

FIG. 5. Graphical representation of the local Nusselt number (Nu) versus the centrifugal Rayleigh number (Ra_ω).

deviation from the linear relationship pertaining to the asymptotic solution ($Nu = 1 + Ra_\omega/24$, according to ref. [8]) becomes significant. The reason for this deviation is the evaluation of Nu in ref. [8] by limiting the asymptotic expansion up to order one. The results of the present solution show that the relationship between Nu and Ra_ω is certainly not linear but closer to a second order polynomial function.

5. CONCLUSIONS

An analytical solution to the non-linear problem of centrifugally generated free convection in a rotating porous box was presented. The assumption that the vertical component of specific flow rate and temperature are independent of x restricts the validity of the solution to small values of the centrifugal Rayleigh number. As a result of increasing the value of Ra_ω the solution to the non-linear algebraic equations for the coefficients a_1 and a_2 leads to physically unacceptable results. This occurs at a value close to $Ra_\omega = 5$. Nevertheless, the results for Ra_ω below this value are reliable and show that the convection affects the temperature profile and the vertical heat flux.

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